Simplification of recourse models by modification of recourse data

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Abstract

We consider modification of the recourse data, consisting of the second-stage parameters and the underlying distribution, as an approximation technique for solving two-stage recourse problems. This approach is applied to several specific classes of recourse problems; in each case, the resulting recourse problem is much easier to solve.

Modification of recourse data is shown to be the common principle behind the approximations which were introduced in previous publications.

Key words: (integer) recourse, approximation

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1 Introduction

Consider the two-stage recourse model with random right-hand side

$$\begin{array}{ll} \min\limits_{x} & cx + Q(x) \\ \text{s.t.} & x \in X := \{x \in \mathbb{R}_+^{n_1} : Ax = b\}, \end{array}$$

with recourse function Q,

$$Q(x) := \mathbb{E}_\omega \left[v(\omega - Tx) \right], \quad x \in \mathbb{R}^{n_1},$$

and second-stage value function v,

$$v(s) := \begin{array}{ll} \min\limits_{y} & qy \\ \text{s.t.} & Wy = s, \qquad s \in \mathbb{R}^m. \\ & y \in Y \end{array}$$

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The distribution of the random right-hand side parameter $\omega \in \mathbb{R}^m$ is assumed to be known; we will denote its cumulative distribution function (cdf) by F, and its probability density function (pdf) by f (if it exists). The set $Y \subset \mathbb{R}^n$ specifies simple bounds and/or integrality restrictions on the second-stage variables y. The vectors and matrices c, A, b, T, q, and W, have conformable dimensions.

Obviously, all characteristic difficulties of such a recourse model are captured by the recourse function Q. Depending on the *recourse structure*, represented by the triple (q, W, Y), and the distribution of ω given by its cdf F, the function Q may or may not have nice mathematical properties and be relatively easy or very difficult to evaluate. For example, if Y specifies integrality restrictions on (some of) the second-stage variables, the function Q is in general non-convex; it is precisely the convexity which underlies all efficient algorithms for solving recourse models with continuous variables.

All essential information about a recourse model can therefore be summarized by the tuple (q, W, Y, F), which we will call the *recourse data*.

If a given recourse problem is difficult to solve, a natural approach is to construct an approximating problem by modifying the recourse data,

$$(q, W, Y, F) \longrightarrow \left(\bar{q}, \bar{W}, \bar{Y}, \bar{F}\right),$$

such that

$$\min_{x \in X} cx + \bar{Q}(x),$$

where $\bar{Q}$ is specified by the recourse data $(\bar{q}, \bar{W}, \bar{Y}, \bar{F})$, is relatively easy to solve.

A well-known example of this approach applies to continuous recourse models with continuously distributed right-hand side vector ω. To avoid the very difficult evaluation of Q, which amounts to evaluating an m-dimensional integral in this case, discrete approximations of the distribution of ω can be constructed. For example, the given instance with $(q, W, \mathbb{R}^n_+, F)$ may be replaced by $(q, W, \mathbb{R}^n_+, F_l)$ and/or $(q, W, \mathbb{R}^n_+, F_u)$, where F_l and F_u are discrete distribution functions, yielding lower and upper bounds of the function Q, respectively. See e.g. the text book [1] for details.

In combination with an algorithm for solving the approximating problem, modification of the recourse data constitutes an algorithm for solving the original recourse problem. In this paper we review the application of this conceptual algorithm to three model types, namely *simple integer recourse*, *complete integer recourse*, and *multiple simple (integer) recourse*. As we will see, in all cases the modification of the problem data involves both the distribution of ω as well as the recourse structure (q, W, Y). In particular, for all three integer recourse model types, the resulting approximation proves to be a continuous recourse problem.

2 Simple integer recourse

The simple integer recourse model, introduced in [11], is characterized by the value function

$$
\begin{aligned}
v(s) := \min_{y} \quad & q^{+} y^{+} + q^{-} y^{-} \\
\text{s.t.} \quad & y^{+} \geq s \\
& y^{-} \geq -s \\
& y = (y^{+}, y^{-}) \in \mathbb{Z}_{+}^{2m}
\end{aligned}
\qquad , \quad s \in \mathbb{R}^{m}.
$$

Like its continuous recourse counterpart, distinguished by the recourse structure

$$
\left(\left[q^{+}, q^{-} \right], \left[I_m, -I_m \right], \mathbb{R}_{+}^{2m} \right) \tag{1}
$$

where I_m is the m-dimensional identity matrix, this value function is separable in s. Assuming that $q^{+} \geq 0$ and $q^{-} \geq 0$,

$$
v(s) = \sum_{i=1}^{m} v_i(s_i), \qquad s \in \mathbb{R}^{m},
$$

with

$$
v_i(s_i) = q_i^{+} \lceil s_i \rceil^{+} + q_i^{-} \lfloor s_i \rfloor^{-},
$$

where $\lceil t \rceil^{+} := \max\{0, \lceil t \rceil\}$ and $\lfloor t \rfloor^{-} := \max\{0, -\lfloor t \rfloor\}$, $t \in \mathbb{R}$. Consequently, the simple integer recourse function Q is separable in the *tender variables* $z := Tx \in \mathbb{R}^{m}$, allowing to restrict the discussion to the one-dimensional case. Thus, for the time being we assume that s and z are scalar variables, and the one-dimensional recourse function Q is

$$
Q(z) = q^{+} \mathbb{E}_{\omega} \left[\lceil \omega - z \rceil^{+} \right] + q^{-} \mathbb{E}_{\omega} \left[\lfloor \omega - z \rfloor^{-} \right], \quad z \in \mathbb{R}, \tag{2}
$$

where ω is a scalar random variable now.

Except for the rounding operations in (2), this is the same formula as the well-known expression for the one-dimensional continuous simple recourse function. Below, this similarity will be exploited to construct an approximation by modification of the recourse data: the resulting problem will be of continuous simple recourse type.

As shown in [11], the simple integer recourse function Q is in general non-convex and even discontinuous in case ω follows a discrete distribution. The following structural result was derived in [8] (under some technical conditions which are not restrictive in practice).

Theorem 2.1 *The simple integer recourse function Q defined in (2) is convex on $\mathbb{R}$ if and only if ω is a continuous random variable whose pdf f satisfies*

$$
f(s) = G(s+1) - G(s), \quad s \in \mathbb{R},
$$

where G is an arbitrary cdf with finite mean value.

In particular, the following special case follows from Theorem 2.1 with G the cdf of a discrete distribution on $\alpha + \mathbb{Z}$, for arbitrary fixed $\alpha \in [0, 1)$.

Corollary 2.1 *Let the pdf f of the random variable ω be piecewise constant on every interval $[\alpha + k, \alpha + k + 1)$, $k \in \mathbb{Z}$, for some fixed $\alpha \in [0, 1)$. Then the function Q is convex on $\mathbb{R}$.*

Corollary 2.1 underlies a class of convex approximations of the simple integer recourse function Q. See [7] for further details.

Definition 2.1 Let ω be a random variable with cdf F. Then, for every fixed $\alpha \in [0, 1)$, the random variable ω_α with piecewise constant pdf

$$f_\alpha(s) := F(\lceil s \rceil_\alpha) - F(\lceil s \rceil_\alpha - 1), \quad s \in \mathbb{R}, \tag{3}$$

where $\lceil t \rceil_\alpha := \lceil t - \alpha \rceil + \alpha$, $t \in \mathbb{R}$, is the round up of t with respect to $\alpha + \mathbb{Z}$, is called an *α-approximation of ω*. Analogously, the convex function Q_α, defined as

$$Q_\alpha(z) = q^+ \mathbb{E}_\omega \left[\lceil \omega_\alpha - z \rceil^+ \right] + q^- \mathbb{E}_\omega \left[\lfloor \omega_\alpha - z \rfloor^- \right], \quad z \in \mathbb{R},$$

is called an *α-approximation of Q*.

For technical reasons, the analysis in [7] is restricted to the case that ω is continuously distributed. Then the following uniform error bound on the approximation of Q by Q_α holds.

Theorem 2.2 *Assume that ω is continuously distributed with a pdf f which is of bounded variation. Then, for $\alpha \in [0, 1)$,*

$$\| Q - Q_\alpha \|_\infty \leq \frac{q^+ + q^-}{4} |\Delta| f,$$

where $|\Delta| f$ is the total variation of f.

Since $|\Delta| f$ tends to be small if the variance of the continuous distribution of ω is not too large, α-approximations provide good convex approximations of Q in practice. For discrete distributions no such performance guarantee exists, but α-approximations may perform well in practice. See also the remarks at the end of this section.

So far, we have seen that a particular transformation of the distribution yields a *convex* approximation, but the approximating problem is still of the simple *integer* recourse type. The following result, obtained in [5], allows a further modification of the recourse data.

Theorem 2.3 *Let $\varphi(s)$, $s \in \mathbb{R}$, be a finite non-linear convex function, and let φ have asymptotes with slopes $-a_1$ as $s \to -\infty$ and a_2 as $s \to \infty$. Then φ is a one-dimensional continuous simple recourse function, given by*

$$\varphi(s) = a_1 \mathbb{E}_\xi \left[(\xi - s)^+ \right] + a_2 \mathbb{E}_\xi \left[(\xi - s)^- \right] + C, \quad s \in \mathbb{R},$$

where C is a known constant and ξ is a random variable with cdf

$$\Phi(t) = \frac{\varphi'_+(t) + a_1}{a_1 + a_2}, \quad t \in \mathbb{R},$$

with φ'_+ the right derivative of the function φ.

Application of Theorem 2.3 to any α-approximation of Q yields the following equivalent representation as a *continuous* simple recourse function.

Corollary 2.2 *Choose $\alpha \in [0, 1)$. Then the α-approximation Q_α satisfies*

$$Q_\alpha(z) = q^+ \mathbb{E}_{\xi^\alpha}\left[(\xi^\alpha - z)^+\right] + q^- \mathbb{E}_{\xi^\alpha}\left[(\xi^\alpha - z)^-\right] + \frac{q^+ q^-}{q^+ + q^-}, \quad z \in \mathbb{R},$$

where ξ^α is a discrete random variable with cdf

$$\Phi_\alpha(t) = \frac{q^+ F(\lceil t \rceil_\alpha - 1) + q^- F(\lceil t \rceil_\alpha)}{q^+ + q^-}, \quad t \in \mathbb{R}.$$

The distribution of ξ^α follows from the formula for the right derivative of Q as derived in [11], which for the function Q_α reads

$$\begin{aligned} (Q_\alpha)'_+(z) &= -q^+ \sum_{k=0}^{\infty} f_\alpha(z + k) + q^- \sum_{k=0}^{\infty} f_\alpha(z - k) \\ &= q^+ \big(F(\lceil z \rceil_\alpha - 1) - 1\big) + q^- F(\lceil z \rceil_\alpha), \quad z \in \mathbb{R}, \end{aligned}$$

where the second expression follows by substitution of (3).

Returning to the full-dimensional model, we conclude the following. Let W_{SIR} denote the recourse matrix of the simple integer recourse problem.

Theorem 2.4 *The simple integer recourse problem with recourse data*

$$\Big(\left[q^+, q^-\right], W_{\text{SIR}}, \mathbb{Z}_+^{2m}, F \Big)$$

can be approximated by the continuous simple recourse problem with recourse data

$$\Big(\left[q^+, q^-\right], \left[I_m, -I_m\right], \mathbb{R}_+^{2m}, \Phi_\alpha \Big),$$

where $\alpha \in [0, 1)^m$ is arbitrary but fixed, and the one-dimensional marginal distributions of Φ_α follow from those of F as specified in Corollary 2.2.

The optimal value of the latter recourse problem needs to be corrected by a known constant.

Together with any algorithm for solving continuous simple recourse problems with discretely distributed right-hand side vector, this modification of the recourse data provides an algorithm for approximately solving simple integer recourse problems. There exist several very efficient special purpose algorithms for continuous simple recourse models, see e.g. [12] or [1].

We conclude the discussion on simple integer recourse models with a short remark on the case with *discrete distributions*. As shown in [6], there exists a fully polynomial algorithm for constructing the convex hull (pointwise largest convex minorant) of the recourse function Q in this case, for which Theorem 2.3 yields the equivalent continuous simple recourse formulation; also in this case the resulting distribution is discrete. Thus, analogous to the case with continuous distributions, modification of the recourse data is the main ingredient of an approximation algorithm for simple integer recourse models with discretely distributed right-hand side vector.

3 Complete integer recourse

Next we consider the complete integer recourse model, which has second-stage value function

$$v(s) := \begin{array}{ll} \min\limits_{y} & qy \\ \text{s.t.} & Wy \geq s \\ & y \in \mathbb{Z}_+^n \end{array}, \qquad s \in \mathbb{R}^m,$$

where the recourse matrix W is such that $v(s) < +\infty$ for all $s \in \mathbb{R}^m$. Note that we use inequalities here to avoid almost sure infeasibility; of course, using slack variables the problem may be stated with equality constraints as in the introduction.

Following the exposition in [15], we will show that also for this general model type convex approximations can be obtained by modification of the recourse data. Again, the approximating recourse problem obtained in this way has continuous second-stage variables and a discretely distributed right-hand side vector.

Throughout, we assume that the recourse is sufficiently expensive so that v is finite on $\mathbb{R}^m$. Since we also assume that $\mathbb{E}_\omega[|\omega_i|] < \infty$, $i = 1, \ldots, m$, it follows that the complete integer recourse function $Q(x) = \mathbb{E}_\omega[v(\omega - Tx)]$ is finite for all $x \in \mathbb{R}^{n_1}$. Finally, we assume that all elements of the recourse matrix W are integral (or rational, so that integrality may be obtained by scaling).

We start out by making the additional assumption that the matrix W is *totally unimodular* (TU). As we will see, this allows to construct the convex hull of the recourse function Q. At the end of this section, we discuss the corresponding results for the general case.

Using that W is a TU matrix, and, moreover, that the right-hand side $s \in \mathbb{R}^m$ can be replaced by $\lceil s \rceil$ since Wy is integral for any $y \in \mathbb{Z}^n$, it follows from strong LP duality that

$$v(s) = \begin{array}{ll} \max\limits_{\lambda} & \lambda \lceil s \rceil \\ \text{s.t.} & \lambda W \leq q, \ \lambda \in \mathbb{R}_+^m. \end{array} \tag{4}$$

Since v is finite, this dual feasible set is a nonempty bounded polyhedron, so that

$$v(s) = \max_{k=1,\ldots,K} \lambda^k \lceil s \rceil, \quad s \in \mathbb{R}^m,$$

where the nonnegative vectors λ^k, $k = 1, \ldots, K$, denote the finitely many extreme points of this set.

Thus we see that v is the pointwise maximum of finitely many round up functions $\lambda^k \lceil s \rceil$, $s \in \mathbb{R}^m$. It turns out that *expected round up functions*, defined for arbitrary fixed $\lambda \in \mathbb{R}_+^m$ as

$$R(z) := \lambda \mathbb{E}_\omega [\lceil \omega - z \rceil], \quad z \in \mathbb{R}^m,$$

provide the key to constructing the convex hull of the recourse function Q. As before, z can be interpreted as a vector of tender variables.

Consider the case that $m = 1$ and $\lambda = 1$. Then by straightforward computation we find that

$$R(z) = \mathbb{E}_\omega\left[\lceil \omega - z \rceil^+\right] - \mathbb{E}_\omega\left[\lfloor \omega - (z-1) \rfloor^-\right], \quad z \in \mathbb{R},$$

at least if ω is continuously distributed. The obvious similarity to the one-dimensional simple integer recourse function (2) correctly suggests that this function R is convex if ω satisfies the assumptions of Corollary 2.1. Indeed, the analogous result can be shown for the general m-dimensional expected round up function, allowing to define α-approximations of this function, similar to the simple integer recourse case.

Definition 3.1 Let $\omega \in \mathbb{R}^m$ be a random vector with arbitrary continuous or discrete distribution, and choose $\alpha = (\alpha_1, \ldots, \alpha_m) \in [0,1)^m$. Define *the α-approximation ω_α* as the random vector with joint pdf f_α that is constant on every hypercube

$$C_\alpha^k := \prod_{i=1}^m (\alpha_i + k_i - 1, \alpha_i + k_i], \quad k \in \mathbb{Z}^m,$$

and such that $\Pr\{\omega_\alpha \in C_\alpha^k\} = \Pr\{\omega \in C_\alpha^k\}$, $k \in \mathbb{Z}^m$.

Analogously, the convex function R_α defined as

$$R_\alpha(z) := \lambda \mathbb{E}_{\omega_\alpha}\left[\lceil \omega_\alpha - z \rceil\right], \quad z \in \mathbb{R}^m,$$

is called an *α-approximation of R*.

Writing, for $z \in \mathbb{R}^m$,

$$\begin{aligned} R_\alpha(z) &= \sum_{i=1}^m \lambda_i \mathbb{E}_{\omega_\alpha}\left[\lceil (\omega_\alpha)_i - z_i \rceil\right] \\ &= \sum_{i=1}^m \lambda_i \left(\mathbb{E}_{\omega_\alpha}\left[\lceil (\omega_\alpha)_i - z_i \rceil^+\right] - \mathbb{E}_{\omega_\alpha}\left[\lfloor (\omega_\alpha)_i - (z_i - 1) \rfloor^-\right]\right), \end{aligned}$$

and applying Corollary 2.2 to each of the individual terms, we obtain the following equivalent representation of R_α.

Lemma 3.1 *For an arbitrary but fixed $\alpha \in [0,1)^m$, let ω_α be an α-approximation of the random vector ω. Then*

$$R_\alpha(z) = \lambda \mathbb{E}_{\xi^\alpha}\left[\xi^\alpha - z\right] = \lambda\left(\mu^\alpha - z\right), \quad z \in \mathbb{R}^m,$$

where $\xi^\alpha = \lceil \omega \rceil_\alpha = \lceil \omega - \alpha \rceil + \alpha$ is a discrete random vector with mean value μ^α and support in $\alpha + \mathbb{Z}^m$, with

$$\Pr\{\xi^\alpha = \alpha + k\} = \Pr\{\omega \in C_\alpha^k\}, \quad k \in \mathbb{Z}^m.$$

Hence, the function R_α is affine with gradient $-\lambda$.

It is not difficult to see that $R_\alpha(z) = R(z)$ for all $z \in \alpha + \mathbb{Z}^m$, and that R_α is neither a lower bound nor an upper bound for R in general. However, observing that $R(z+k) = R(z) - \lambda k$, $k \in \mathbb{Z}^m$, for every z, we see that $R(z) + \lambda z$ is a periodic function, which repeats itself on every set C_α^k. Hence, defining

$$\alpha^\star \in \operatorname{argmin} \left\{ R(z) + \lambda z : z \in [0,1)^m \right\}, \tag{5}$$

the affine function $R_{\alpha^\star}$ is a lower bound for R, which is sharp at every $z \in \alpha^\star + \mathbb{Z}^m$. It follows that $R_{\alpha^\star}$ is the convex hull of R. As shown in [15], this optimal shift parameter $\alpha^\star$ can be computed in almost all practically relevant cases.

Now we are ready to state the main results of this section, namely on convex approximations of the recourse function Q and the corresponding modifications of the recourse data. Although no explicit reference is made to results for the function R above, they are stated in terms of optimal α-approximations and the equivalent representations as introduced for the latter function. The formal proofs (see [15]) which are omitted here, do rely extensively on the above results.

Theorem 3.1 *Consider the integer recourse expected value function Q, defined as*

$$Q(x) = \mathbb{E}_\omega \left[\min_y qy : Wy \geq \omega - Tx,\ y \in \mathbb{Z}_+^n \right], \quad x \in \mathbb{R}^{n_1}, \tag{6}$$

and the continuous recourse expected value function $Q_{\alpha^\star}$, defined as

$$Q_{\alpha^\star}(x) = \mathbb{E}_{\xi^{\alpha^\star}} \left[\min_y qy : Wy \geq \xi^{\alpha^\star} - Tx,\ y \in \mathbb{R}_+^n \right], \quad x \in \mathbb{R}^{n_1}, \tag{7}$$

with $\alpha^\star$ and $\xi^{\alpha^\star}$ as defined in (5) and Lemma 3.1.

Under the assumptions stated above, in particular that W is totally unimodular, the function $Q_{\alpha^\star}$ is the convex hull of Q if the matrix T is of full row rank. If $\operatorname{rank} T < m$, then $Q_{\alpha^\star}$ is a lower bound for Q.

The condition on the row rank of the matrix T follows from the results in [4].

Next we drop the assumption that W is a TU matrix (but still assume that it is integral). In this case (4) holds only with inequality, so that $Q_{\alpha^\star}$ still provides a lower bound for the recourse function Q, but it will not be equal to the convex hull. However, in the sense of Theorem 3.2 below, $Q_{\alpha^\star}$ is a strictly better convex approximation than Q^{LP} which is obtained using the LP relaxation of the second-stage problem,

$$Q^{\text{LP}}(x) := \mathbb{E}_\omega \left[\min_y \left\{ qy : Wy \geq \omega - Tx,\ y \in \mathbb{R}_+^n \right\} \right], \quad x \in \mathbb{R}^{n_1}. \tag{8}$$

Theorem 3.2 *Consider the functions $Q_{\alpha^\star}$ and Q^{LP}, defined by (7) and (8) respectively, which both are convex lower bounds for the integer recourse expected value function Q, defined by (6).*

(a) $Q_{\alpha^\star} \geq Q^{\text{LP}}$.

(b) *Assume*

(i) $q \geq 0$*, so that* 0 *is a trivial lower bound for* v *and* Q*;*

(ii) *there exists a subset* L *of* $\mathbb{Z}^m$ *such that the distribution of* ω *with support* Ω *satisfies* $\Omega \subset \bigcup_{l \in L} \{\omega : \omega \leq \alpha^\star + l\}$ *and* $\Pr\{\omega < \alpha^\star + l \mid \omega \in C^l_{\alpha^\star}\} > 0$ *for all* $l \in L$*.*

Then the function $Q_{\alpha^\star}$ *is a strictly better convex approximation of* Q *than* Q^{LP}*, in the sense that* $Q(x) > 0$ *implies* $Q_{\alpha^\star}(x) > Q^{\text{LP}}(x)$*.*

For example, condition (ii) of Theorem 3.2 is satisfied if ω follows a non-degenerated continuous distribution.

Summarizing, we propose the following modification of the recourse data in order to approximately solve complete integer recourse models.

Theorem 3.3 *The complete integer recourse problem with recourse data*

$$\left(q, W, \mathbb{Z}^n_+, F\right)$$

can be approximated by the continuous complete recourse problem with recourse data

$$\left(q, W, \mathbb{R}^n_+, \Phi_{\alpha^\star}\right),$$

where $\alpha^\star \in [0, 1)^m$ *is defined by (5), and* $\Phi_{\alpha^\star}$ *is the cdf of the discrete random vector* $\xi^{\alpha^\star}$ *as defined in Lemma 3.1.*

In case the recourse matrix W is TU, the resulting continuous recourse problem may be solved to obtain an approximate solution (exact if it belongs to the interior of X) of the original model. In the general case, the approximation will often not be good enough for this purpose; instead, it may be used to provide a lower bound as used by several existing algorithms for complete integer recourse problems such as *integer L-shaped* [10], *stochastic branch-and-bound* [13], and *structured enumeration* [14] (see [9] for a survey). Not only does our approach yield a better approximation in the sense of Theorem 3.2, but because of the discrete distribution of $\xi^{\alpha^\star}$ it is also easier to compute, in particular if ω is continuously distributed.

4 Multiple simple recourse

The third class of models to which we apply our recourse data modification approach consists of multiple simple recourse models, both with continuous and integer second-stage variables. The continuous version of this model was introduced in [3] to allow for piecewise linear penalty costs for shortages and surpluses with respect to individual constraints. Thus, it is a generalization of the well-known simple recourse model, which assigns linear penalty costs to such deviations.

Using recourse data modification, we will show that continuous multiple simple recourse (MSR) models can be restated as simple recourse (SR) models. Consequently, continuous MSR models can be solved using the available efficient algorithms for SR models. Exploiting the results of Section 2, similar results hold for α-approximations of integer MSR models. For motivating examples, proofs, and further details, we refer to [16].

Like in the simple recourse case, the value function of the MSR model is separable. To avoid unnecessary notational burden, we restrict the detailed presentation to the one-dimensional case.

For $s \in \mathbb{R}$, the one-dimensional value function of the continuous MSR model is defined as

$$
\begin{aligned}
v(s) := \min_y \quad & \sum_{k=1}^{K} \left(q_k^+ y_k^+ + q_k^- y_k^-\right) \\
\text{s.t.} \quad & \sum_{k=1}^{K} y_k^+ - \sum_{k=1}^{K} y_k^- = s \\
& \left.\begin{aligned} y_k^+ &\le u_k - u_{k-1}, \\ y_k^- &\le l_k - l_{k-1}, \end{aligned}\right\} \quad k = 1, \ldots, K-1 \\
& y = (y^+, y^-) \in \mathbb{R}_+^{2K}
\end{aligned}
$$

with $u_0 = l_0 = 0$ and

$$
\begin{array}{llll}
0 \le & q_1^+ \le \ldots \le & q_{K-1}^+ \le & q_K^+ \\
0 \le & u_1 \le \ldots \le & u_{K-1} & \\
0 \le & q_1^- \le \ldots \le & q_{K-1}^- \le & q_K^- \\
0 \le & l_1 \le \ldots \le & l_{K-1} &
\end{array}
\tag{9}
$$

It easy to see that this function v is piecewise linear, and that it is convex due to the conditions on the cost coefficients.

For later reference, we note that the recourse matrix W_{MSR} of the m-dimensional MSR second-stage problem is given by

$$
W_{\text{MSR}} = \begin{pmatrix} e_1 & & & & -e_1 & & & \\ & e_2 & & & & -e_2 & & \\ & & \ddots & & & & \ddots & \\ & & & e_m & & & & -e_m \end{pmatrix},
$$

where e_i is a K_i-vector of ones, $i = 1, \ldots, m$. The feasible set of recourse actions is

$$
Y_{\text{MSR}} = \prod_{i=1}^{m} \left\{ (y_i^+, y_i^-) \in \mathbb{R}_+^{2K_i} : \begin{aligned} y_{ik}^+ &\le u_{ik} - u_{i,k-1}, \\ y_{ik}^- &\le l_{ik} - l_{i,k-1}, \end{aligned} \quad k = 1, \ldots, K_i - 1 \right\},
$$

and the corresponding recourse cost vector

$$
q_{\text{MSR}} = \left(q_{11}^+, q_{12}^+, \ldots, q_{mK_m}^+, q_{11}^-, q_{12}^-, \ldots, q_{mK_m}^-\right)
$$

satisfies conditions as specified in (9). The assumption that y_i^+ and y_i^- have the same number of components K_i, $i = 1, \ldots, m$, is without loss of generality.

Note that for $K_i = 1, i = 1, \ldots, m$, the MSR recourse structure reduces to the SR recourse structure (1), confirming that the MSR model is a generalization of the SR model.

Returning to the one-dimensional case, the following closed form for the MSR value function can be derived by straightforward computation.

$$v(s) = \sum_{k=0}^{K-1} \Big[\left(q_{k+1}^{+} - q_{k}^{+}\right)(s - u_k)^{+} + \left(q_{k+1}^{-} - q_{k}^{-}\right)(s + l_k)^{-} \Big], \quad s \in \mathbb{R},$$

where we conveniently define $q_0^+ = q_0^- = 0$. Taking the expectation of $v(\omega - z)$, where z denotes a tender variable as before, we find that for $z \in \mathbb{R}$

$$Q(z) = \sum_{k=0}^{K-1} \Big[\left(q_{k+1}^{+} - q_{k}^{+}\right) G(z + u_k) + \left(q_{k+1}^{-} - q_{k}^{-}\right) H(z - l_k) \Big], \tag{10}$$

with, for $s \in \mathbb{R}$,

$$G(s) := \mathbb{E}_\omega \left[(\omega - s)^{+}\right] \quad \text{and} \quad H(s) := \mathbb{E}_\omega \left[(\omega - s)^{-}\right],$$

is a closed form for the (one-dimensional) MSR expected value function.

Properties of the *expected surplus function* G and the *expected shortage function* H are well known from the analysis of SR models. In particular, for $s \in \mathbb{R}$,

$$G'_{+}(s) := F(s) - 1 \quad \text{and} \quad H'_{+}(s) := F(s),$$

where F is the cdf of the random variable ω. Moreover, assuming that ω has mean value μ, the function $G(s)$ has asymptotes $\mu - s$ as $s \to -\infty$ and 0 as $s \to \infty$; $H(s)$ has asymptotes 0 as $s \to -\infty$ and $s - \mu$ as $s \to \infty$. Using this information, we apply Theorem 2.3 to obtain the following equivalent representation of the convex function Q.

Corollary 4.1 *Consider the one-dimensional MSR function Q, given in closed form by (10). Then*

$$Q(z) = q_K^{+} \mathbb{E}_\xi \left[(\xi - z)^{+}\right] + q_K^{-} \mathbb{E}_\xi \left[(\xi - z)^{-}\right] - C, \quad z \in \mathbb{R},$$

where ξ is a random variable with cdf V,

$$V(t) = \frac{\displaystyle\sum_{k=0}^{K-1} \Big[\left(q_{k+1}^{+} - q_{k}^{+}\right) F(t + u_k) + \left(q_{k+1}^{-} - q_{k}^{-}\right) F(t - l_k) \Big]}{q_K^{+} + q_K^{-}}, \quad t \in \mathbb{R},$$

and F is the cdf of ω. The constant C is given by

$$C = \frac{q_K^{+} \displaystyle\sum_{k=1}^{K-1} \left(q_{k+1}^{-} - q_{k}^{-}\right) l_k + q_K^{-} \sum_{k=1}^{K-1} \left(q_{k+1}^{+} - q_{k}^{+}\right) u_k}{q_K^{+} + q_K^{-}}.$$

Thus, the MSR function Q can be represented as an SR function, whose underlying distribution is explicitly known in terms of the problem parameters. In particular, if the random variable ω in the MSR formulation is discrete, then so is the random variable ξ in the SR representation. In this case the distribution of ξ can be specified directly, without reference to the distribution function of ω, see [16].

We summarize our results on continuous MSR models in the following theorem, stated in terms of the full-dimensional problem.

Theorem 4.1 *The multiple simple recourse problem with recourse data*

$$\left(q_{\text{MSR}}, W_{\text{MSR}}, Y_{\text{MSR}}, F\right)$$

is equivalent to the simple recourse problem with recourse data

$$\left([q^+_{\text{SR}}, q^-_{\text{SR}}], [I_m, -I_m], \mathbb{R}^{2m}_+, V\right),$$

where $q^+_{\text{SR}} := (q^+_{1K_1}, \dots, q^+_{mK_m})$, $q^-_{\text{SR}} := (q^-_{1K_1}, \dots, q^-_{mK_m})$, *and the one-dimensional marginal distributions of* V *follow from those of* F *as specified in Corollary 4.1.*

The optimal value of the latter problem needs to be corrected by a known constant.

The implied algorithm for solving MSR problems, consisting of this modification of the recourse data and subsequently solving the resulting SR problem, is implemented as Mscr2Scr 1.0 (Multiple simple continuous recourse to Simple continuous recourse, M.H. van der Vlerk and J. Mayer, 2001) in the model management system SLP-IOR [2]. The current version of Mscr2Scr is restricted to problems with discrete random variables.

Finally, we consider modification of the recourse data for the MSR model with integer second-stage variables, which we will denote as multiple simple integer recourse (MSIR) with recourse structure $(q_{\text{MSIR}}, W_{\text{MSIR}}, Y_{\text{MSIR}})$. As before, we first concentrate on the one-dimensional case.

For $s \in \mathbb{R}$, the one-dimensional MSIR value function is defined as

$$\begin{aligned} v(s) := \min_y \quad & \sum_{k=1}^K \left(q^+_k y^+_k + q^-_k y^-_k\right) \\ \text{s.t.} \quad & \sum_{k=1}^K y^+_k \geq s, \quad \sum_{k=1}^K y^-_k \geq -s \\ & \left.\begin{aligned} y^+_k &\leq u_k - u_{k-1}, \\ y^-_k &\leq l_k - l_{k-1}, \end{aligned}\right. \quad k = 1, \dots, K-1 \\ & y = (y^+, y^-) \in \mathbb{Z}^{2K}_+ \end{aligned}$$

with $u_0 = l_0 = 0$, integer vectors u and l, and the elements of q^+, u, q^-, and l satisfying the same monotonicity assumptions (9) as in the continuous recourse setting. Observe that if $K = 1$, then v equals the simple integer recourse value function.

Like for the continuous MSR model, it is easy to obtain a closed form expression for the MSIR value function v. Subsequently, taking the expectation of $v(\omega - z)$ yields the following expression for the (one-dimensional) MSIR function Q: for $z \in \mathbb{R}$,

$$Q(z) = \sum_{k=0}^{K-1} \Big[\big(q_{k+1}^{+} - q_{k}^{+}\big)\, \mathcal{G}(z + u_k) + \big(q_{k+1}^{-} - q_{k}^{-}\big)\, \mathcal{H}(z - l_k) \Big],$$

where $q_0^{+} = q_0^{-} = 0$ and

$$\mathcal{G}(z) := \mathbb{E}_\omega \left[\lceil \omega - z \rceil^{+} \right] \quad \text{and} \quad \mathcal{H}(z) := \mathbb{E}_\omega \left[\lfloor \omega - z \rfloor^{-} \right].$$

On replacing $\mathcal{G}$ and $\mathcal{H}$ by their α-approximations for any fixed $\alpha \in [0, 1)$, see Definition 2.1, we obtain an α-approximation Q_α of the MSIR function Q. For $z \in \mathbb{R}$,

$$Q_\alpha(z) := \sum_{k=0}^{K-1} \Big[\big(q_{k+1}^{+} - q_{k}^{+}\big)\, \mathcal{G}_\alpha(z + u_k) + \big(q_{k+1}^{-} - q_{k}^{-}\big)\, \mathcal{H}_\alpha(z - l_k) \Big]. \tag{11}$$

Such convex approximations Q_α can be defined for any distribution of ω; however, a non-trivial error bound only holds if the distribution is continuous. Theorem 2.3 yields the following equivalent representation of Q_α.

Corollary 4.2 *For a fixed but arbitrary $\alpha \in [0, 1)$, consider the α-approxima-tion Q_α of the MSIR expected value function Q as defined in (11). Then*

$$Q_\alpha(z) = q_K^{+} \mathbb{E}_{\xi^\alpha} \left[(\xi^\alpha - z)^{+} \right] + q_K^{-} \mathbb{E}_{\xi^\alpha} \left[(\xi^\alpha - z)^{-} \right] + D, \quad z \in \mathbb{R},$$

where ξ^α is a random variable with cdf V_α: for $t \in \mathbb{R}$,

$$V_\alpha(t) = \frac{\displaystyle\sum_{k=0}^{K-1} \Big[\big(q_{k+1}^{+} - q_{k}^{+}\big)\, F(\lfloor t \rfloor_\alpha + u_k) + \big(q_{k+1}^{-} - q_{k}^{-}\big)\, F(\lfloor t \rfloor_\alpha + 1 - l_k) \Big]}{q_K^{+} + q_K^{-}},$$

where F is the cdf of ω and $\lfloor t \rfloor_\alpha := \lfloor t - \alpha \rfloor + \alpha$ is the round down of t with respect to $\alpha + \mathbb{Z}$. That is, ξ^α is discretely distributed, with support contained in $\alpha + \mathbb{Z}$ and probabilities

$$\begin{aligned} \Pr\{\xi^\alpha = \alpha + j\} \quad = \quad & \frac{1}{q_K^{+} + q_K^{-}} \sum_{k=0}^{K-1} \Big[\big(q_{k+1}^{+} - q_{k}^{+}\big) \Pr\{\omega \in \Omega_\alpha^{j} + u_k\} \\ & + \ \big(q_{k+1}^{-} - q_{k}^{-}\big) \Pr\{\omega \in \Omega_\alpha^{j+1} - l_k\} \Big], \quad j \in \mathbb{Z}, \end{aligned}$$

where $\Omega_\alpha^{j} := (\alpha + j - 1, \alpha + j]$.

The constant D is given by

$$D = \frac{q_K^{+}\, q_K^{-}}{q_K^{+} + q_K^{-}} - C,$$

where C is the constant given in Corollary 4.1.

The implications for the full-dimensional MSIR model can be summarized as follows.

Theorem 4.2 *The multiple simple integer recourse model with recourse data*

$$\left(q_{\mathrm{MSIR}},\, W_{\mathrm{MSIR}},\, Y_{\mathrm{MSIR}},\, F\right)$$

can be approximated by the continuous simple recourse problem with recourse data

$$\left([q^+_{\mathrm{SR}}, q^-_{\mathrm{SR}}], [I_m, -I_m], \mathbb{R}^{2m}_+, V_\alpha\right),$$

where $\alpha \in [0, 1)^m$ *is arbitrary but fixed, and the one-dimensional marginal distributions of* V_α *follow from those of* F *as specified in Corollary 4.2.*

The optimal value of the latter recourse problem needs to be corrected by a known constant.

Thus, to approximately solve an MSIR problem, it suffices to solve an explicitly given continuous simple recourse problem with discretely distributed right-hand side parameters.

5 Summary and concluding remarks

We have shown that modification of recourse data is a fruitful approach, at least for the three classes of recourse models which we considered. Indeed, it allows integer versions of (multiple) simple and complete recourse models to be approximated by their continuous recourse counterparts, simply by applying a suitable transformation to the underlying distribution of the random right-hand side parameters. In the same way, continuous multiple simple recourse models are reduced to ordinary simple recourse models.

This paper stresses the obvious fact that there are two components which above all determine how hard a given recourse problem is: the recourse structure, and the characteristics of the underlying distribution. For example, integrality restrictions and continuous distributions are both complicating factors in general. However, as we have seen, if the recourse structure and the distribution harmonize with each other, then the corresponding recourse problem turns out to be 'nice' after all. Hence, the potential power of the concept of recourse data modification is that it focuses on the *interaction* of the recourse data constituents.

In future research we will investigate the use of recourse data modification for specific (integer) recourse models, also in the multistage setting. Moreover, initial results suggest that a similar approach is suitable to obtain approximations of certain chance-constrained problems.

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